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Regularized non-Abelian chiral Jacobian factor at finite temperature

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Abstract. The regularized non-Abelian chiral Jacobian factor in the path-integral formulation of fermions interacting with background vector and axial vector fields is investigated at finite temperature in arbitrary even dimensions. After separating out the terms with normal parity, it is shown that the self-consistent non-Abelian chiral Jacobian is the same at $T = 0$ and $T \neq 0$.

It is well known that the current anomaly of a quantized fermion field with gauge coupling is connected with the ultraviolet properties of the theory, and it is expected that the formal result of an anomaly should be the same at $T = 0$ and $T \neq 0$. The temperature independence of an anomaly was first noticed by Dolan and Jackiw (1974) in their pioneering work on the symmetry behaviour of field theories at finite temperature. They showed that the vacuum-polarization tensor of two-dimensional QED at finite temperature coincides with the one at zero temperature and that the mass of the gauge boson does not change with temperature T. The same conclusion was also reached later in four-dimensional QCD by observing that the finite-temperature corrections to the axial vector divergence of the real-time three-point Green function vanishes to all orders (Itoyama and Mueller 1983). The non-perturbative proof of the temperature independence of the anomaly was also given by some authors. In the context of QED_2 the methods of ζ-function regularization (Reuter and Dittrich 1985) and the derivative expansion (Das and Karev 1987) have been used within the path-integral formalism. Recently, Liu and Ni (1988) also made a proof for the four-dimensional Abelian chiral anomaly in this formalism, but they failed to generalize their proof to the non-Abelian case, especially with axial gauge coupling.

In this paper, we are going to illustrate that in order to generalize the proof of the temperature independence of the non-Abelian chiral anomaly in the path-integral framework, the exponential of the Jacobian factor induced by an infinitesimal chiral rotation must be divided into real and imaginary parts; after regularization only the imaginary part yields a finite self-consistent result with abnormal parity which we can prove to be temperature independent, while the real part yields terms with normal parity which are regularization parameter dependent and also temperature dependent, but these terms can be cancelled out by adding counterterms to the original Lagrangian.

In order to make our proof more simple and straightforward, we start by recapitulating the derivation of the regularized non-Abelian chiral Jacobian factor at zero temperature. The generating functional for fermion fields interacting with background

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gauge fields including γ_{2n+1} coupling defined in $2n$-dimensional Euclidean space is

$$Z(V, A) = \int \mathrm{d}\bar{\psi}\, \mathrm{d}\psi \exp\left(-\int \mathrm{d}^{2n}x\, \bar{\psi}\mathrm{i}\not{D}\psi\right) \tag{1}$$

where $\mathrm{i}\not{D}$ is the Dirac operator

$$\mathrm{i}\not{D} = \mathrm{i}\gamma^{\mu} D\mu = \mathrm{i}\gamma^{\mu}(\partial\mu + V\mu + \gamma_{2n+1}A\mu)$$

$$V\mu = \mathrm{i}\, V\mu_a T^a \qquad A\mu = \mathrm{i}A\mu_a T^a$$

with $\gamma_{2n+1} = \mathrm{i}^n \gamma_1\gamma_2 \ldots \gamma_{2n}$, $\{\gamma_\mu, \gamma_\nu\} = 2\delta\mu\nu$ and T^a are the Hermitian generators of group $\mathrm{U}(N)$; $V\mu_a$, $A\mu_a$ are the real background fields. Following Fujikawa's (1985) prescription, one expands ψ and $\bar{\psi}$ in terms of Grassmann variables a_n and $\overline{b_n}$:

$$\psi = \sum a_n \phi_n \qquad \bar{\psi} = \sum \varphi_n^+ \overline{b_n}$$

where ϕ_n and φ_n^+ are the separate right and left eigenfunctions of $\mathrm{i}\not{D}$

$$(\mathrm{i}\not{D})\phi_n = \lambda_n \phi_n \qquad (\mathrm{i}\not{D})^+ \varphi_n = \lambda_n^* \varphi_n$$

with the normalization

$$\int \mathrm{d}^{2n}x\, \varphi_n^+ \phi_n = \delta_{mn}.$$

The action, therefore, simplifies to

$$\int \mathrm{d}^{2n}x\, \varphi_n^+ \mathrm{i}\not{D}\phi_n = \sum \overline{b_n}\, a_n \lambda_n.$$

Note that λ_n should be considered as a complex number. Interpreting the fermion measure $\mathrm{d}\bar{\psi}\, \mathrm{d}\psi$ as $\Pi\, \mathrm{d}\overline{b_n}\, \mathrm{d}a_n$, the generating functional (1) becomes

$$Z(V, A) = \operatorname{Det} \mathrm{i}\not{D} = \prod \lambda_n. \tag{2}$$

Under a chiral transformation specified by an infinitesimal real local function $\alpha(x) = \alpha_a(x)T^a$,

$$\psi \to \psi' = \exp(\mathrm{i}\alpha\gamma_{2n+1})\gamma \qquad \bar{\psi} \to \bar{\psi}' = \bar{\psi}\exp(\mathrm{i}\alpha\gamma_{2n+1})$$

the Jacobian factor of the non-invariant path-integral measure can be expressed as the ratio of two determinants,

$$J^{-1} = \exp(\mathrm{i}\delta\Gamma) = \frac{\operatorname{Det} \mathrm{i}\not{D}'}{\operatorname{Det} \mathrm{i}\not{D}} = \frac{\Pi\, \lambda_n'}{\Pi\, \lambda_n} \tag{3}$$

where

$$\mathrm{i}\not{D}' = \mathrm{i}\not{D} + \{\mathrm{i}\alpha\gamma_{2n+1}, \mathrm{i}\not{D}\} + \mathrm{O}(\alpha^2).$$

Letting $\delta\lambda = \lambda' - \lambda$, and treating $\delta(\mathrm{i}\not{D}) = \{\mathrm{i}\alpha\gamma_{2n+1}, \mathrm{i}\not{D}\}$ as a perturbation, one has

$$\delta\lambda = \int \mathrm{d}x\, \varphi_n^+ \delta(\mathrm{i}\not{D})\phi_n = \mathrm{i}2\lambda_n \int \mathrm{d}x(\varphi_n^+ \alpha\gamma_{2n+1}\phi_n) \tag{4}$$

and

$$\mathrm{i}\delta\vartheta_n = \ln\frac{\lambda_n'}{\lambda_n} = \mathrm{i}2\int \mathrm{d}x(\varphi_n^+ \alpha\gamma_{2n+1}\phi_n). \tag{5}$$

It is easy to check that the expression (5) is also applicable to the zero modes: this can be realized by introducing an infinitesimal term $\varepsilon\gamma_{2n+1}$ into the Dirac operator at the very beginning and taking the limit $\varepsilon \to 0$ (Wang and Ni 1987). Therefore, $\delta\Gamma$ in (3) is

$$\delta\Gamma = \sum_m \delta\vartheta_m = 2\sum_m \int \mathrm{d}^{2n}x(\varphi_m^+ \alpha\gamma_{2n+1}\phi_m). \tag{6}$$

This implies that the exponential of the Jacobian factor can be understood as the accumulated effect of a dilation and of an argument change of all the diagonal elements λ_n induced by an infinitesimal chiral rotation. In the special case of $A = 0$, as discussed by Liu and Ni (1988), the Dirac operator $\mathrm{i}\not{D}$ is Hermitian, $\varphi_n = \phi_n$, $\delta\vartheta_n$ turns out to be real (corresponding to the pure argument change of λ_n induced by the chiral rotation), and $\delta\Gamma$ can be interpreted as the summation of this change over all the eigenvalues. In the general case of a fermion field theory with axial gauge coupling, $A \neq 0$, the Dirac operator is non-Hermitian. In this case λ_n should be understood as a point in the complex plane, and $\delta\vartheta_n$ should be decomposed into real and imaginary parts in which only the real part is related to the pure argument change of λ_n induced by the chiral rotation. The real and imaginary parts of $\delta\Gamma$ can be separated as

$$\delta\Gamma = \delta\Gamma_+ + \delta\Gamma_-$$

with

$$\begin{aligned}\delta\Gamma_\pm &= \tfrac{1}{2}\sum_m (\delta\vartheta_m \pm \delta\vartheta_m^*) \\ &= \sum_m \int \mathrm{d}^{2n}x(\varphi_m^+ \alpha\gamma_{2n+1}\phi_m \pm \phi_m^+ \alpha\gamma_{2n+1}\varphi_m).\end{aligned} \tag{7}$$

To obtain meaningful results from this infinite sum, a regularization procedure must be followed in order to suppress the contributions from high modes (Fujikawa 1980). In the case of the non-Abelian gauge coupling, one must also impose the Wess–Zumino (wz) self-consistent condition in this procedure. As already explained by Alvarez-Gaumé and Ginsparg (1984), the wz condition is a one-cocycle condition which simply implies that if the transformation is along an infinitesimal closed loop in the Lie algebra space of the gauge symmetry group, all the diagonal elements λ_n of the Dirac operator will also trace out a closed loop in the complex plane. One easily finds that the argument change of all the non-zero modes along this closed loop is zero, while the argument change of the zero mode is $\pm 2\pi$, where the sign depends on the chirality of the zero mode. In the sense of keeping the non-trivial contributions of the zero modes within the constraint of the wz condition, we adopt the M-independent regularization scheme (Wang and Ni 1987) which is found to be particularly convenient in generalizing to the case of finite temperature, namely

$$\begin{aligned}\delta\Gamma_\pm^{\mathrm{reg}} = \int \mathrm{d}x \Bigg[&\left(\sum_m \varphi_m^+ \alpha\gamma_{2n+1}(\mathrm{i}M)(\mathrm{i}M + \lambda_m)^{-1}\phi_m\right) \\ &\pm \left(\sum_m \phi_m^+ \alpha\gamma_{2n+1}(\mathrm{i}M)(\mathrm{i}M + \lambda_m^*)^{-1}\varphi_m\right)\Bigg]_{M\,\mathrm{indep}} \\ = \int \mathrm{d}x \Bigg[&\left(\sum_m \varphi_m^+ \alpha\gamma_{2n+1}(\mathrm{i}M)(\mathrm{i}M + \mathrm{i}\not{D})^{-1}\phi_m\right) \\ &\pm \left(\sum_m \phi_m^+ \alpha\gamma_{2n+1}(\mathrm{i}M)(\mathrm{i}M + \mathrm{i}\tilde{\not{D}})^{-1}\varphi_m\right)\Bigg]_{M\,\mathrm{indep}}\end{aligned} \tag{8}$$

where $\mathrm{i}\tilde{\not{D}} = (\mathrm{i}\not{D})^+$.

Now denoting $\{\Phi_n\}$ and $\{\Psi_n\}$ as the orthonormal complete bases of the Hermitian operators $(\mathrm{i}\not{D})^+(\mathrm{i}\not{D})$ and $(\mathrm{i}\not{D})(\mathrm{i}\not{D})^+$, respectively, one finds that although these bases are different from $\{\phi_n\}$ and $\{\varphi_n\}$ of the non-Hermitian Dirac operators $(\mathrm{i}\not{D})$ and $(\mathrm{i}\not{D})^+$, the subspaces spanned by the zero modes can be selected as the same. In the sense that only the contribution of the zero modes is preserved, one can change the bases by rewriting (8) as

$$\begin{aligned}\delta\Gamma^{\mathrm{reg}}_{\pm} &= \int \mathrm{d}x \Bigg[\Bigg(\sum_m \Phi^+_m \alpha\gamma_{2n+1}(\mathrm{i}M)(\mathrm{i}M+\mathrm{i}\not{D})^{-1}\Phi_m\Bigg) \\ &\qquad \pm\Bigg(\sum_m \Psi^+_m \alpha\gamma_{2n+1}(\mathrm{i}M)(\mathrm{i}M+\mathrm{i}\tilde{\not{D}})^{-1}\Psi_m\Bigg)\Bigg]_{M\,\mathrm{indep}} \\ &= \lim_{y\to x}\mathrm{Tr}\Bigg[\Bigg(\alpha\gamma_{2n+1}\frac{M}{M+\not{D}}\delta^{2n}(x-y)\Bigg) \\ &\qquad \pm\Bigg(\alpha\gamma_{2n+1}\frac{M}{M+\tilde{\not{D}}}\delta^{2n}(x-y)\Bigg)\Bigg]_{M\,\mathrm{indep}}. \end{aligned} \tag{9}$$

In a detailed evaluation one can expand the trace in (9) in the following way:

$$\begin{aligned}\lim_{y\to x}\mathrm{Tr}&\Bigg(\alpha\gamma_{2n+1}\frac{M}{M+\not{D}}\delta^{2n}(x-y)\Bigg) \\ &= (2\pi)^{-2n}\int \mathrm{d}^{2n}k\,\mathrm{Tr}\Bigg(\alpha\gamma_{2n+1}(M-\mathrm{i}k-\tilde{\not{D}})\frac{1}{(M+\mathrm{i}k+\not{D})(M-\mathrm{i}k-\tilde{\not{D}})}\Bigg) \\ &= (2\pi)^{-2n}\int \mathrm{d}^{2n}k\,\mathrm{Tr}\Bigg(\alpha\gamma_{2n+1}(M-\mathrm{i}k-\tilde{\not{D}})\frac{1}{M^2+k^2-Q}\Bigg) \\ &= (2\pi)^{-2n}\int \mathrm{d}^{2n}k\,\mathrm{Tr}\Bigg(\alpha\gamma_{2n+1}(M-\mathrm{i}k-\tilde{\not{D}})\sum_l Q^l\frac{1}{(M^2+k^2)^{l+1}}\Bigg) \end{aligned} \tag{10}$$

where

$$Q = \tilde{\not{D}}\not{D} - \mathrm{i}2MA\gamma_{2n+1} + \mathrm{i}2k_\mu D_\mu .$$

A similar procedure can be applied to the second trace in (9). The detailed evaluation of the trace in (10) is tedious but can be performed following a similar procedure to that in previous works (Gipson 1986, Wang and Ni 1987, Wang 1988). The result is

$$\begin{aligned}\delta\Gamma^{\mathrm{reg}}_{+} = \frac{1}{(4\pi)^n n!}\mathrm{Tr}\Bigg[&\alpha\gamma_{2n+1}\sum_{m=0} B(m+1,n+1) \\ &\times\Bigg(\frac{1}{(2m)!}\frac{\partial^{2m}}{\partial M^{2m}}\sum_{l_1+l_2=m+n}(Q_+^{l_1}Q_-^{l_2}+Q_-^{l_1}Q_+^{l_2})\Bigg)_{M=0}\Bigg]\end{aligned} \tag{11}$$

in which

$$Q_\pm = F^V_{\mu\nu}\sigma_{\mu\nu} \pm F^A_{\mu\nu}\sigma_{\mu\nu} + 2MA\gamma_{2n+1}$$

$$B(m+1,n+1) = m!\,n!/(m+n+1) \qquad \sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/4$$

while $\delta\Gamma^{\mathrm{reg}}_{-} = 0$.

From (11) we can easily read off the self-consistent chiral anomaly of the fermion theory defined in arbitrary even dimensions.

Based on the above description, the proof of the temperature independence of the regularized self-consistent non-Abelian Jacobian becomes very straightforward. The main strategy in our proof is to separate out the finite temperature contributions in (10). In the case of finite temperature, $T=\beta^{-1}\neq 0$, the generating functional is

$$Z_\beta(V,A)=\int \mathrm{d}\bar{\psi}\,\mathrm{d}\psi\,\exp\left(-\int_0^\beta \mathrm{d}x_0\int \mathrm{d}^{2n-1}x\,\bar{\psi}\mathrm{i}\not{D}\psi\right). \tag{12}$$

Here the path integral has to be defined over the space of functions with periodic or antiperiodic conditions imposed in the Euclidean time variable x_0:

$$\psi(x_0+\beta,\boldsymbol{x})=-\psi(x_0,\boldsymbol{x})\qquad \bar{\psi}(x_0+\beta,\boldsymbol{x})=-\bar{\psi}(x_0,\boldsymbol{x})\qquad A_\mu(x_0+\beta,\boldsymbol{x})=A_\mu(x_0,\boldsymbol{x}).$$

Under an infinitesimal chiral transformation specified by an infinitesimal periodic function $\alpha(x_0+\beta,\boldsymbol{x})=\alpha(x_0,\boldsymbol{x})$ the Jacobian factor induced is

$$J^{-1}(\beta)=\exp\{\mathrm{i}\delta\Gamma(\beta)\}=\frac{\mathrm{Det}_\beta\,\mathrm{i}\not{D}'}{\mathrm{Det}_\beta\,\mathrm{i}\not{D}}=\frac{\Pi\,\lambda_n'(\beta)}{\Pi\,\lambda_n(\beta)}$$

where $\delta\Gamma(\beta)$ can be expressed as

$$\delta\Gamma(\beta)=\sum_m\delta\vartheta_m(\beta)=2\sum_m\int_0^\beta \mathrm{d}x_0\int \mathrm{d}^{2n-1}x(\varphi_m^+\alpha\gamma_{2n+1}\phi_m). \tag{13}$$

Almost the same procedure as used at zero temperature leads to the analogue of (9), namely

$$\delta\Gamma_\pm^{\mathrm{reg}}=\lim_{y\to x}\mathrm{Tr}\left[\left(\alpha\gamma_{2n+1}\frac{M}{M+\not{D}}\delta^{2n}(x-y)\right)\pm\left(\alpha\gamma_{2n+1}\frac{M}{M+\tilde{\not{D}}}\delta^{2n}(x-y)\right)\right]_{M\ \mathrm{indep}}. \tag{14}$$

Since ϕ_n, φ_n and Φ_n, Ψ_n are all antiperiodic, the Fourier representation of the δ function which was obtained via the completeness relation of a set of eigenfunctions should be understood as

$$\delta(x-y)=\frac{1}{\beta}\sum_{-\infty}^{\infty}\exp\left(\mathrm{i}\frac{\pi}{\beta}(2n+1)(x_0-y_0)\right)(2\pi)^{-(2n-1)}\int \mathrm{d}^{2n-1}k\,\exp(-\mathrm{i}k(\boldsymbol{x}-\boldsymbol{y})). \tag{15}$$

Therefore, the expansion of the first trace in (14) becomes

$$\begin{aligned}
&\lim_{y\to x}\mathrm{Tr}\left(\alpha\gamma_{2n+1}\frac{M}{M+\not{D}}\delta^{2n}(x-y)\right)\\
&\quad=\beta^{-1}(2\pi)^{-(2n-1)}\sum_{n=-\infty}^{+\infty}\int \mathrm{d}^{2n-1}k\\
&\qquad\times\mathrm{Tr}\left(\alpha\gamma_{2n+1}(M+\mathrm{i}\omega_n\gamma_0-\mathrm{i}\boldsymbol{k\gamma}-\tilde{D})\sum_l Q^l(E_M^2+\omega_n^2)^{-(l+1)}\right)\\
&\quad=\beta^{-1}(2\pi)^{-(2n-1)}\sum_{n=-\infty}^{+\infty}\int \mathrm{d}^{2n-1}k\,\mathrm{Tr}\left(\alpha\gamma_{2n+1}\sum_l G(\omega_n)(E_M^2+\omega_n^2)^{-(l+1)}\right)
\end{aligned} \tag{16}$$

where

$$\omega_n = (2n+1)\pi\beta^{-1} \qquad E_M = \boldsymbol{k}^2 + M^2 \tag{17a}$$

$$Q(\omega_n) = \tilde{\not{D}}\not{D} - \mathrm{i}2MA\gamma_{2n+1} + \mathrm{i}2\boldsymbol{kD} - \mathrm{i}2\omega_n D_0 \tag{17b}$$

$$G(\omega_n) = (M + \mathrm{i}\omega_n\gamma_0 - \mathrm{i}\boldsymbol{k\gamma} - \tilde{\not{D}})Q^l(\omega_n). \tag{17c}$$

The sum over the Fermi–Dirac frequency in (16) can be treated as follows. First, one notices that the odd part of $G(\omega_n)$ does not contribute after summation over n, while the even part of $G(\omega_n)$ can be expanded as

$$\frac{G(x)^{\text{even}}}{(a^2+x^2)^{-l+1}} = \sum_{0\leqslant s\leqslant l+1} \frac{b_s}{(a^2+x^2)^{-s}} = \sum_{0\leqslant s\leqslant l+1} \frac{(-1)^{s-1}}{(s-1)!}\left(\frac{\partial}{\partial x^2}\right)^{s-1} \frac{b_s}{(a^2+x^2)} \tag{18}$$

where the b_n are x-independent coefficients. By rewriting,

$$\beta^{-1}\sum_{n=-\infty}^{+\infty} G(\omega_n)(E_M^2+\omega_n^2)^{-s}$$

$$= \beta^{-1} \sum_{0\leqslant s\leqslant l+1} \frac{(-1)^{s-1}}{(s-1)!}\left(\frac{\partial}{\partial E_M^2}\right)^{s-1} \sum_{n=-\infty}^{+\infty} b_s(E_M^2+\omega_n^2)^{-1} \tag{19}$$

and applying the formula

$$\beta^{-1}\sum_{n=-\infty}^{+\infty} (E_M^2+\omega_n^2)^{-1} = \frac{1}{2E_M}\left(1 - \frac{1}{\mathrm{e}^{\beta E_M}+1}\right) \tag{20}$$

one finds

$$\beta^{-1}\sum_{n=-\infty}^{+\infty} G(\omega_n)(E_M^2+\omega_n^2)^{-s}$$

$$= \sum_{0\leqslant s\leqslant l+1} \frac{(-1)^{s-1}}{(s-1)!}\left(\frac{\partial}{\partial E_M^2}\right)^{s-1} b_s\left(\frac{1}{2E_M} - \frac{1}{E_M}\frac{1}{\mathrm{e}^{\beta E_M}+1}\right)$$

$$= \sum_{0\leqslant s\leqslant l+1} \frac{(-1)^{s-1}}{(s-1)!}\left(\frac{\partial}{\partial E_M^2}\right)^{s-1} \left(\int_{-\infty}^{+\infty} \frac{\mathrm{d}k_0}{2\pi}\frac{b_s}{E_M^2+k_0^2} - \frac{1}{E_M}\frac{b_s}{\mathrm{e}^{\beta E_M}+1}\right)$$

$$= \int_{-\infty}^{+\infty} \frac{\mathrm{d}k_0}{2\pi}\frac{G(k_0)}{(E_M^2+k_0^2)^{l+1}} - \sum_{0\leqslant s\leqslant l+1} \frac{(-1)^{s-1}}{(s-1)!}\left(\frac{\partial}{\partial E^2}\right)^{s-1} \frac{1}{E_m}\frac{b_s}{\mathrm{e}^{\beta E_M}+1}. \tag{21}$$

Here the equality (18) is used, and the fact that the integral relating to the odd part of $G(k_0)$ does not contribute is also noticed.

Thus the trace shown in (16) at finite temperature is divided into two parts, namely

$$\lim_{y\to x} \mathrm{Tr}\left(\alpha\gamma_{2n+1}\frac{M}{M+\not{D}}\delta^{2n}(x-y)\right)$$

$$= (2\pi)^{-2n}\int \mathrm{d}^{2n}k\, \mathrm{Tr}\left(\alpha\gamma_{2n+1}(M-\mathrm{i}\tilde{\not{D}})\sum_l Q^l \frac{1}{(M^2+k^2)^{l+1}}\right)$$

$$-\beta^{-1}(2\pi)^{-(2n-1)}\int \mathrm{d}^{2n-1}k \sum_{0\leqslant s\leqslant l+1} \frac{(-1)^{s-1}}{(s-1)!}\left(\frac{\partial}{\partial E_M^2}\right)^{s-1}\left(\frac{1}{E_M}\frac{b_s}{\mathrm{e}^{\beta E_M}+1}\right). \tag{22}$$

The first part is an integral which takes the same form as that at zero temperature as shown in (10). The second part contains the terms arising from finite-temperature effects. By expanding

$$\frac{(-1)^{s-1}}{(s-1)!}\left(\frac{\partial}{\partial E_M^2}\right)^{s-1}\left(\frac{1}{E_M}\frac{b_s}{e^{\beta E_M}+1}\right)$$

$$=\frac{2(-1)^s}{\beta(s-1)!}\left(\frac{\partial}{\partial E_M^2}\right)^s \ln(1+e^{-\beta E_M})$$

$$=\sum_{j=1}^{\infty}\frac{2(-1)^{s+j}}{\beta(s-1)!j}\left(\frac{\partial}{\partial E_M^2}\right)^s e^{-j\beta E_M} \tag{23}$$

one finds that every term in the second part is M dependent and does not contribute to the anomaly after M-independent regularization. The same procedure can be applied to the second trace in (14) and yields the same conclusion. Therefore we have proved that the regularized self-consistent non-Abelian chiral Jacobian factor $\delta\Gamma_+^{\text{reg}}$ at finite temperature coincides with the one at zero temperature.

Note that in the evaluation of the regularized chiral Jacobian factor one can also take the large-M limit instead of using the M-independent regularization procedure. In the large-M limit, the second term of (22) is exponentially small and $\delta\Gamma_+^{\text{reg}}$ remains the same as at zero temperature. This means that the M-independent regularization is equivalent to that of the large-M limit in the sense of evaluating $\delta\Gamma_+^{\text{reg}}$ which is the self-consistency 'abnormal parity' contribution to the anomaly (i.e. the terms with antisymmetric symbol $\varepsilon_{\mu_1\nu_1\ldots\mu_n\nu_n}$ after taking the trace of γ matrices (Hu *et al* 1984)). But in the evaluation of $\delta\Gamma_-^{\text{reg}}$, the result using the large-M limit procedure does not vanish; one easily finds that it is M dependent and also temperature dependent. It will lead to the 'normal parity naive anomaly' contributions (i.e. the terms without the antisymmetric symbol $\varepsilon_{\mu_1\nu_1\ldots\mu_n\nu_n}$), which terms can be subtracted by adding the counter-terms in the original Lagrangian (Fujikawa 1985). Therefore in the proof of the temperature independence of non-Abelian chiral anomaly in a theory with axial gauge coupling, one should first separate out $\delta\Gamma_-$, the 'normal parity contributions', from the exponential of the Jacobian factor before taking the large-M limit in the regularization procedure (or equivalently the small-t limit in the paper of Liu and Ni (1988)).

Furthermore, since the trace of an odd number of γ matrices is zero, one has the equality

$$\lim_{y\to x}\operatorname{Tr}\left(\alpha\gamma_{2n+1}\frac{M}{M+\not{D}}\,\delta^{2n}(x-y)\right)=\lim_{y\to x}\operatorname{Tr}\left(\alpha\gamma_{2n+1}\frac{M^2}{M^2+(\mathrm{i}\not{D})^2}\,\delta^{2n}(x-y)\right). \tag{24}$$

This implies that the regulator chosen here is of the Pauli-Villars type, which is a particular form of the $f(\not{D}^2/M^2)$ and $f(\tilde{\not{D}}^2/M^2)$ suggested by Fujikawa. Since the result of $\delta\Gamma_+^{\text{reg}}$ is independent of the detailed form of the regulator $f(\not{D}^2/M^2)$ and $f(\tilde{\not{D}}^2/M^2)$, the temperature independence of $\delta\Gamma_+^{\text{reg}}$ can also be proved by inserting any regulator which satisfies the condition $f(0)=1, f'(\infty)=f''(\infty)=\ldots=0$ (Fujikawa 1980, 1985).

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